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The Existence of Generalized Eigenfunctions In Underwater Acoustics

Environmental and Tactical Support Systems Department

Naval Underwater Systems Center
Newport, Rhode Island • New London, Connecticut

PREFACE

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An example is given of a horizontally stratified acoustic waveguide that supports generalized eigenfunctions in addition to the usual eigenfunctions or normal modes. Generalized eigenfunctions occur when the characteristic equation has a zero with a multiplicity greater than one. Both eigenfunctions and generalized eigenfunctions are required to provide a complete representation of some functions. The separation of variables solution for a point source in a waveguide, based on the usual eigenfunctions or normal modes, is not valid for this example.

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INTRODUCTION

The acoustic field due to a point source in a horizontally stratified waveguide can be expressed as a contour integral.[1] This contour follows a path in the complex plane that separates the singularities of the depth- and range-separated Green's functions. Because it is assumed that the depth-separated problem is finite, the depth-separated Green's function will have poles, but no branch-cut singularities. The poles of the Green's function are the zeros of the characteristic equation and are the same as the eigenvalues. When the contour is enclosed around the poles, the solution can be expressed as the sum of the residues at the poles. If the depth-separated problem is self-adjoint, then the poles of the Green's function are simple, the zeros of the characteristic function have a multiplicity equal to one, and the residue series is the usual normal mode solution.[2]

If the depth-separated problem is non-self-adjoint, the poles of the Green's function may not be simple, the zeros of the characteristic function can have a multiplicity greater than one, and the residue series assumes a more complicated form.[3] The eigenfunctions must be augmented with the generalized eigenfunctions (called associated functions in Naimark[3]) to yield an expansion theorem. An example of this situation, with separated boundary conditions, is shown in a useful book by Friedman.[4] His example corresponds to an isovelocity waveguide bounded by a free surface and a special complex impedance bottom boundary condition. Similar examples

occur in acoustic ducts with admittance boundary conditions.[5]

Non-self-adjoint problems occur in underwater acoustics when the square of the wavenumber is given a complex value to allow for volume attenuation. This report shows that complex wavenumbers can give rise to multiple eigenvalues and generalized eigenfunctions in the same way as complex impedance and admittance boundary conditions give rise to multiple eigenvalues.

THEORY

Consider a waveguide where the square of the wavenumber $k^2(x) = g\,x + h$ is a complex-valued linear function of the depth x for $0 \le x \le H$. The waveguide is assumed to be bounded by a free surface and bottom. The homogeneous depth-separated problem is given by the complex Sturm-Liouville boundary value (eigenvalue) problem

$$\frac{d^2u}{dx^2} + \left[g\,x + h - \lambda\right]u = 0\,, \qquad (1a)$$

$$u(0) = 0\,, \qquad (1b)$$

$$u(H) = 0\,, \qquad (1c)$$

where λ is the complex separation parameter. A nonzero function $\phi_n(x)$ that satisfies equation (1) is an eigenfunction and the corresponding λ_n is an eigenvalue. The Green's function $G(x,s,\lambda)$ of the depth-separated problem satisfies the nonhomogeneous counterpart of equation (1a) with a delta function source at $x = s$. The Green's function also satisfies the boundary conditions in equations (1b) and (1c).

Suppose that the differential operator L is defined by

$$L\,u = \frac{d^2u}{dx^2} + \left[g\,x + h\right]u \qquad (2)$$

for functions satisfying the boundary conditions in equations (1b) and (1c). The eigenfunction corresponding to λ_n satisfies $(L-\lambda_n)\phi_n = 0$. A generalized eigenfunction (of rank 2) corresponding to λ_n is a nonzero

function ψ_n that satisfies $(L-\lambda_n)^2\psi_n = 0$, but $(L-\lambda_n)\psi_n \neq 0$. The Green's function satisfies $(L-\lambda)G = \delta(x-s)$.

Equation (1a) can be solved with Airy functions after the linear change of variables is made:

$$z(x) = -g^{-2/3} [\, g\, x + (h - \lambda) \,] \,. \tag{3}$$

Then equation (1) becomes

$$\frac{d^2w}{dz^2} - z\, w = 0 \,, \tag{4a}$$

$$w(z_1) = 0 \,, \tag{4b}$$

$$w(z_2) = 0 \,, \tag{4c}$$

where $w(z) = u(x)$, $z_1 = z(0)$, and $z_2 = z(H)$. Grosjean and De Meyer[6] have shown that there exist distinct complex numbers z_1 and z_2 and a corresponding nonzero solution of equation (4) that, in addition, satisfies

$$\int_{z_1}^{z_2} w^2(z)\, dz = 0 \,, \tag{5}$$

where w is determined uniquely up to an arbitrary nonzero multiplicative factor.

The solution of Grosjean and De Meyer[6] is transformed into a solution of equation (1) with

$$g = \frac{(z_1 - z_2)^3}{H^3} \quad , \tag{6a}$$

$$g^{-2/3} = \frac{H^2}{(z_1 - z_2)^2} \quad , \tag{6b}$$

and

$$h - \lambda = \frac{-z_1 \, (z_1 - z_2)^2}{H^2} \quad . \tag{6c}$$

The resulting solution $\phi_n(x) = w[z(x)]$ is an eigenfunction of equation (1) and, in addition, satisfies

$$\int_0^H \phi_n^2(x) \; dx \; = \; 0 \quad . \tag{7}$$

Once h is chosen, the $\lambda = \lambda_n$ in equation (6c) is the eigenvalue corresponding to ϕ_n.

If λ_n were a simple pole of the Green's function, the residue at λ_n would be[2]

$$\left[\int_0^H \phi_n^2(x) \; dx \right]^{-1} \phi_n(x) \; \phi_n(s) \quad , \tag{8}$$

which is impossible in view of equation (7). Thus, λ_n is not a simple pole of the Green's function, and a generalized eigenfunction must exist that

corresponds to λ_n. Equation (7) can also be used, following the argument in Ince,[7] to show that the derivative of the characteristic function, at λ_n, is zero. Consequently, λ_n is not a simple zero of the characteristic function.

EXAMPLE

Let $z_1 = \rho\, e^{-i\,\pi/3}$ and $z_2 = \rho\, e^{i\,\pi}$ be a pair of values given by Grosjean and De Meyer,[6] where ρ is a positive root of

$$J_{1/3}(2/3\ \rho^{3/2}) = 0 \tag{9}$$

and $J_{1/3}$ is the Bessel function of order 1/3. Based on equation (6a), the gradient is purely imaginary and is given by

$$g = -8\,\rho^3 \frac{\sin^3(\pi/3)}{H^3}\ i\ . \tag{10}$$

From equation (6c), the quantity $h - \lambda$ is given by

$$h - \lambda = 4\,\rho^3 \frac{\sin^2(\pi/3)}{H^2}\ [\,\cos(\pi/3) + \sin(\pi/3)\ i\,]\ . \tag{11}$$

Recall that $k^2(x) = g\,x + h$ and $k^2(0) = h$. Let h be chosen as

$$h = \frac{(2\pi\ f)^2}{c^2} + 8\,\rho^3 \frac{\sin^3(\pi/3)}{H^2}\ i\ , \tag{12}$$

where c is the nominal sound speed in m/s and f is the frequency in hertz. The real part of h is the same as the real wavenumber squared. The imaginary part of h is designed to make $k^2(H) = (2\pi\ f)^2/c^2$. The multiple eigenvalue λ_n is

$$\lambda_n = \frac{(2\pi\ f)^2}{c^2} - 4\rho^3 \frac{\sin^2(\pi/3)\ \cos(\pi/3)}{H^2} + 4\rho^3 \frac{\sin^3(\pi/3)}{H^2}\ i\ . \tag{13}$$

The horizontal wavenumber k_n corresponding to λ_n is $k_n = (\lambda_n)^{1/2}$, where the square root has a positive imaginary part. The complex phase velocity is defined by $c_n = (2\pi f)/ k_n$. The eigenvalues occur in a strip (see the appendix) in the complex λ plane defined by $| \operatorname{Im} \lambda | \leq a$ and $\operatorname{Re} \lambda < b$, where

$$a = 8\rho^3 \frac{\sin^3(\pi/3)}{H^2} \tag{14a}$$

and

$$b = \frac{(2\pi f)^2}{c^2} . \tag{14b}$$

The multiple eigenvalue in equation (13) is in the middle of this strip.

The characteristic function will be constructed, and its behavior at the multiple eigenvalue will be investigated. A solution of the differential equation in equation (1a) that satisfies the boundary condition $u(0, \lambda) = 0$ in equation (1b) is given by

$$u(x,\lambda) = \operatorname{Ai}[z(0,\lambda)] \operatorname{Ai}[e^{\theta i}z(x,\lambda)] - \operatorname{Ai}[z(x,\lambda)] \operatorname{Ai}[e^{\theta i}z(0,\lambda)] , \tag{15}$$

where $\theta = 2\pi/3$ and $z(x, \lambda) = z(x)$ is defined by equation (3). The cube root taken in equation (6b) places the arguments of the Airy functions in the lower half z-plane, and the pair of functions $\operatorname{Ai}[z]$ and $\operatorname{Ai}[e^{\theta i}z]$ are the appropriate linear independent set for this region. The Airy functions can be computed with the algorithm of Schulten, Anderson, and Gordon.[8] The eigenvalues are determined by the roots of the characteristic equation $u(H, \lambda) = 0$ obtained from equation (1c). The eigenvalues are the zeros of the characteristic function $F(\lambda)$, where

$$F(\lambda) = Ai[z(0,\lambda)]\ Ai[e^{\theta i}z(H,\lambda)] - Ai[z(H,\lambda)]\ Ai[e^{\theta i}z(0,\lambda)] \ . \qquad (16)$$

A specific example will now be constructed. Let ρ be the smallest positive root of equation (9) computed with[9]

$$\rho = \left[\ (3/2)\ \ (2.9025862) \ \right]^{2/3} = \ 2.6663527 \ . \qquad (17)$$

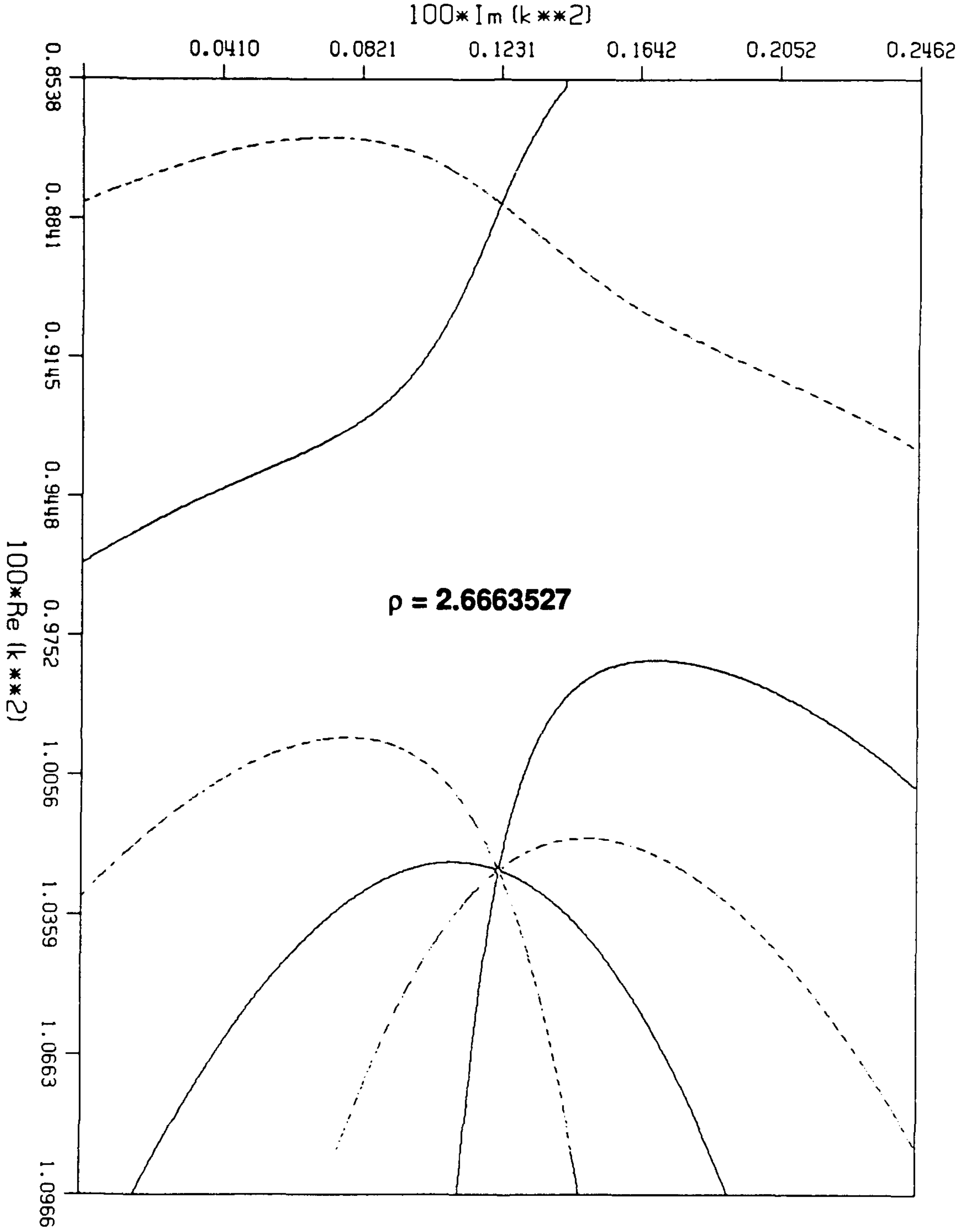
Suppose that the sound speed is $c = 1500$ m/s, the frequency is $f = 25$ Hz, and the thickness of the waveguide is $H = 200$ m. The imaginary part of $k^2(x)$ is linearly decreasing from a maximum at the surface to zero at the bottom. The maximum attenuation α, computed with

$$\alpha = \frac{20\ \log_{10}(e)\ c}{f} \ \mathrm{Im}\left[k(0)\right] \ , \qquad (18)$$

is 6.0896820 dB per wavelength, or about 0.1 dB per meter.

Figure 1 is a plot of the zero-crossing curves of the real and imaginary parts of the characteristic function $F(\lambda)$ in equation (16). The curves are plotted in a rectangular region in the complex $k^2 = \lambda$ plane. The $\mathrm{Re}F(\lambda) = 0$ curves are solid and the $\mathrm{Im}F(\lambda) = 0$ curves are broken. The rectangle coincides with the strip defined by equation (14), except that Re λ is bounded below by $(2\pi f)^2/C^2$, where $C = 1700$ m/s is a maximum phase velocity. The real part of the phase velocity corresponding to the multiple eigenvalue in equation (13) is 1542.8188 m/s. The multiple eigenvalue is contained in the rectangle.

The zero-crossing curves of the real and imaginary parts intersect at a zero of the characteristic function. The principle of the argument

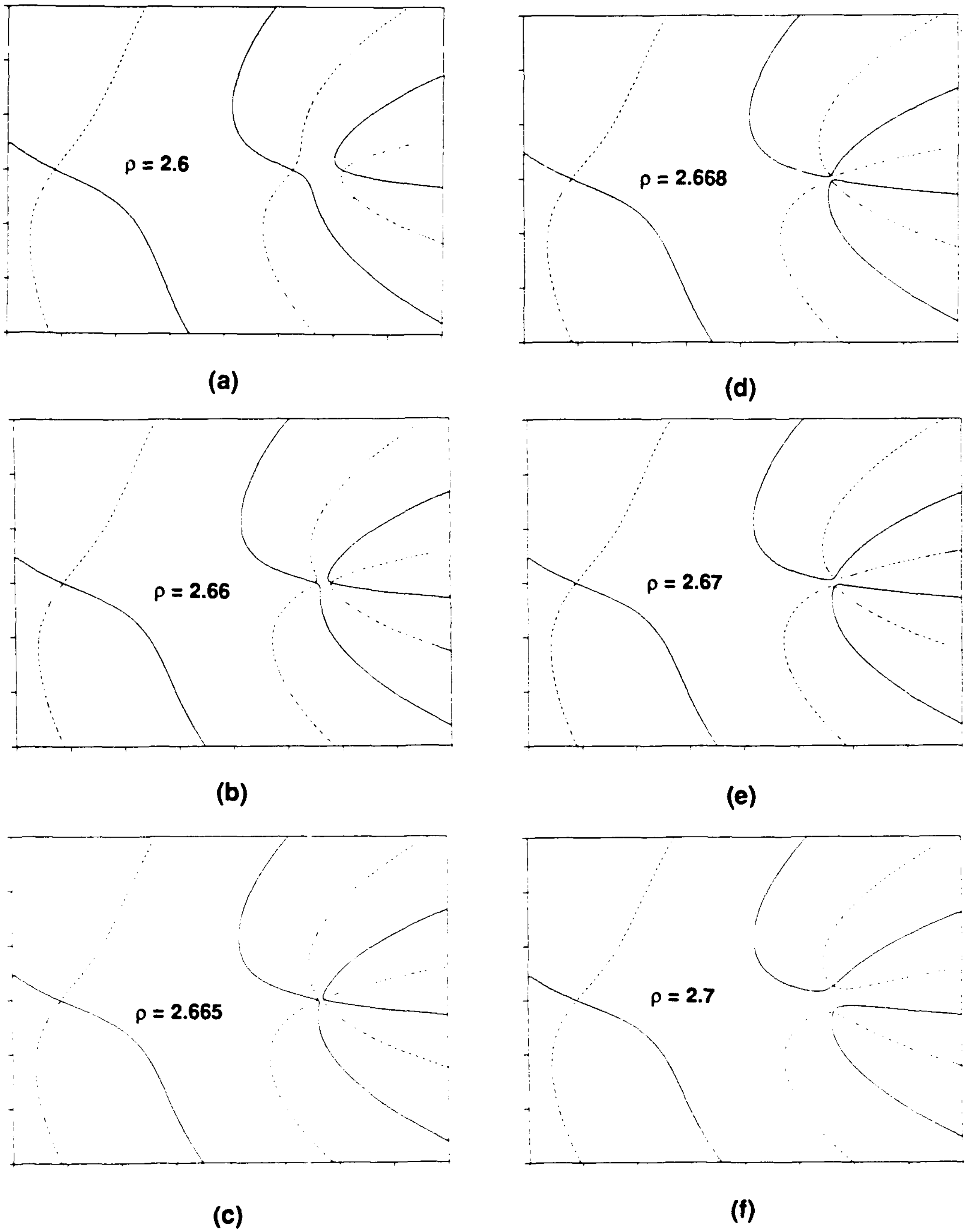


Note: The solid curves are the zeros of the real part and the broken curves are the zeros of the imaginary part. The plot is for a region in the complex $k^2 = \lambda$ plane bounded, in physical units, by the phase velocities 1700 and 1500 m/s and attenuation values 0.0 and 6.089682 dB/wavelength. The axes are annotated with 100 times the real and imaginary parts of $k^2 = \lambda$. The characteristic function of the Example has a double zero in this region of the complex plane.

Figure 1. The Zero-Crossing Curves Plotted for the Real and Imaginary Parts of the Characteristic Function of the Example

implies that two curves intersect at a simple zero and four curves intersect at a double zero. It is apparent from figure 1 that the first eigenvalue, with the smallest phase velocity, is a double zero of the characteristic function. There will be a generalized eigenfunction of rank 2 corresponding to the first eigenvalue. The second eigenvalue is a simple zero of the characteristic function.

Evidence of the double zero can also be seen by considering a sequence of values of the parameter ρ that surrounds the value in equation (17). These values correspond to waveguides with different attenuation gradients. The plots shown in figure 2 are similar to the plot in figure 1, except that ρ has been given the values 2.6, 2.66, 2.665, 2.668, 2.67, and 2.7. In figure 2(a), with $\rho = 2.6$, there are three simple zeros. In figure 2(c), with $\rho = 2.665$, the first two zeros have *moved close to each other and are aligned horizontally.* In figure 2(d), with $\rho = 2.668$, the first two zeros are still close, but now they are aligned vertically. Finally, in figure 2(f), with $\rho = 2.7$, there are again three well-separated simple zeros. The change in alignment occurs at the value of ρ corresponding to the double zero. A movie made from a large number of plots like those in figure 2, with a fine sampling of ρ, would show that the first two zeros coalesce and then separate again.



Note: The parameter ρ has been given the values 2.6, 2.66, 2.665, 2.668, 2.67, and 2.7. The sequence proceeds from top to bottom and left to right. The double zero occurs between the bottom of the first column and the top of the second column. The parameter ρ determines the attenuation gradient in the waveguide.

Figure 2. A Sequence of Plots for Values of the Parameter ρ Surrounding the Value Corresponding to the Double Zero

CONCLUSIONS

The usual separation of variables (normal mode) solution for a point source in the waveguide of the Example in the previous section is not valid. This is because the usual normal mode solution assumes that the poles of the Green's function are simple and that the residues at the poles can be computed by equation (8). The Green's function in the Example has a double pole, which involves a more complicated formula[3] to compute the residue. Thus, the normal mode solution must be modified to include both eigenfunctions and generalized eigenfunctions. A solution can also be obtained with an alternate technique like the fast field program,[10] which computes the contour integral solution directly without using residue theory.

It is clear from the derivation of the Example that there are many such elementary examples. The amount of attenuation required to produce the double pole in the Example is quite substantial. When the amount of attenuation is very limited, it has been shown that all the poles are simple.[11] It remains to be determined if double poles must be treated systematically in normal mode computer codes operating in more realistic underwater acoustic environments.

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APPENDIX

BOUNDS FOR THE LOCATION OF THE COMPLEX EIGENVALUES

This appendix derives two inequalities that bound the location of the complex eigenvalues of the depth-separated wave equation. Consider the differential equation

$$p(x) \frac{d}{dx}\left[\frac{1}{p(x)} \frac{du}{dx}\right] + \left[q(x) - \lambda\right] u = 0 , \qquad \text{(A-1)}$$

where $q(x) = r(x) + i\, s(x)$ is a complex-valued function and $p(x)$ is a positive real-valued function on the interval $0 \le x \le H$. It is not initially assumed that u satisfies any specific boundary conditions. Let the superscript * stand for complex conjugation. If equation (A-1) is multiplied by u^* and the sum and difference are formed with the complex conjugate of the resulting equation, then it follows that

$$u^* \frac{d}{dx}\left[\frac{1}{p(x)} \frac{du}{dx}\right] \pm u \frac{d}{dx}\left[\frac{1}{p(x)} \frac{du^*}{dx}\right]$$

$$= \left[(\lambda \pm \lambda^*) - (q(x) \pm q^*(x))\right] \frac{u u^*}{p(x)} . \qquad \text{(A-2)}$$

Equation (A-2) is integrated by parts on the interval [0,H] to yield

$$\left[\frac{u^*}{p(x)} \frac{du}{dx} \pm \frac{u}{p(x)} \frac{du^*}{dx}\right]_0^H$$

$$- \int_0^H \frac{1}{p(x)} \left[\frac{du^*}{dx} \frac{du}{dx} \pm \frac{du}{dx} \frac{du^*}{dx}\right] dx \qquad \text{(A-3)}$$

$$= \int_0^H \left[(\lambda \pm \lambda^*) - (q(x) \pm q^*(x))\right] \frac{u u^*}{p(x)}\, dx .$$

If $u = \phi_n$ is an eigenfunction and $\lambda = \lambda_n$ is the corresponding eigenvalue, then the boundary conditions at 0 and H can be used to eliminate the first term on the left-hand side of equation (A-3), and it becomes

$$\int_0^H \frac{1}{p(x)} \left[\left| \frac{d\phi_n}{dx} \right|^2 \pm \left| \frac{d\phi_n}{dx} \right|^2 \right] dx =$$

$$\int_0^H \left[(q(x) \pm q^*(x)) - (\lambda_n \pm \lambda_n^*) \right] \frac{|\phi_n|^2}{p(x)} dx , \qquad \text{(A-4)}$$

where the identity $| z |^2 = z z^*$ has been used.

Taking the plus sign, equation (A-4) yields

$$\int_0^H \frac{1}{p(x)} \left| \frac{d\phi_n}{dx} \right|^2 dx =$$

$$\int_0^H r(x) \frac{|\phi_n|^2}{p(x)} dx - \operatorname{Re} \lambda_n \int_0^H \frac{|\phi_n|^2}{p(x)} dx . \qquad \text{(A-5)}$$

Because the integral on the left-hand side of equation (A-5) is positive, it follows that

$$\operatorname{Re} \lambda_n < \left[\int_0^H \frac{|\phi_n|^2}{p(x)} dx \right]^{-1} \int_0^H r(x) \frac{|\phi_n|^2}{p(x)} dx . \qquad \text{(A-6)}$$

Let $r_{max} = \max\{r(x): 0 \le x \le H\}$. Replacing r(x) in the integral in equation (A-6) with its maximum gives the strict inequality

$$\mathrm{Re}\,\lambda_n < r_{max} \ . \tag{A-7}$$

Taking the minus sign in equation (A-4) gives the equation

$$\mathrm{Im}\,\lambda_n = \left[\int_0^H \frac{|\phi_n|^2}{p(x)}\,dx\right]^{-1} \int_0^H s(x)\,\frac{|\phi_n|^2}{p(x)}\,dx \ . \tag{A-8}$$

Let $s_{min} = \min\{s(x): 0 \le x \le H\}$ and $s_{max} = \max\{s(x): 0 \le x \le H\}$. Replacing s(x) in the integral in equation (A-8) with its minimum and maximum gives the inequality

$$s_{min} \le \mathrm{Im}\,\lambda_n \le s_{max} \ . \tag{A-9}$$

In the depth-separated wave equation, $p(x) = \rho(x)$ is density, which is positive, and $q(x) = k^2(x)$ is the square of the wavenumber. Thus, Re λ_n is less than the maximum of $r(x) = \mathrm{Re}\,k^2(x)$, which is determined by the minimum sound speed. The imaginary part of $k^2(x)$, which causes attenuation, is non-negative. Hence, Im λ_n is non-negative and less than or equal to the maximum of $s(x) = \mathrm{Im}\,k^2(x)$. This places λ_n in an infinite half-strip in the upper complex $\lambda = k^2$ plane.

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